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LETTER TO THE EDITOR

Epidemic model with immunisation

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Abstract. Simulations of an epidemic process with imperfect immunisation on a triangular lattice examine the percolating-non-percolating phase transition. The fractal dimensionalities d_f of the set of immune sites and the set of infected sites just above the percolation threshold are analysed. d_f for infections exhibits a sharp drop above the tricritical point. Evidence is given for the fact that the threshold value of the reinfection probability is the same for infection probability $p = 1$ as for no-immunisation.

Models of growth processes have been used to describe a wide range of dynamic problems involving spreading [1-11]. The growth of random clusters on a lattice has, under a variety of circumstances, been likened to an epidemic process, whereby a site which has been infected may infect adjacent sites. In one version, the forest fire model [5], the cluster of sites is equivalent to bond percolation. Other growth models are truly kinetic in that the development of the infected system depends on past history. For instance, the case where an infected site returns to its original condition is equivalent to directed percolation. In many situations, there is no simple percolation eqivalent, but the question of whether the infection process continues indefinitely or not is an important issue.

In the epidemic model, an infected site at time t can infect a neighbouring site at time $t+1$, say, with probability p. If a site can be infected only once, we have the forest fire model (or bond percolation). In this model, an infinite cluster of burnt sites will be formed if $p > p_c$, the percolation threshold. Immunisation of sites is introduced into the infection process by allowing an infected site to recover (the site is then termed immune). Immunisation is perfect if the probability of subsequent re-infection r of a previously infected site by an infected neighbour is zero. This is also equivalent to the forest fire model. If the probability of re-infection r is non-zero, the immunisation process is imperfect. Now, a site which has not yet been infected (susceptible) is still subject to infection with probability p. So, whether infections take place indefinitely, i.e. whether percolation occurs, depends on both the infection probability p and the reinfection probability r.

At any given time, there are four types of sites: (i) susceptible, (ii) infected, (iii) immune and (iv) uninfected. Type-(iv) sites differ from type-(i) or type-(iii) sites in that they have no neighbouring infected sites. In this general epidemic model, the recovery mechanism means that a site may repeatedly become infected and then immune. There is no permanent cluster of immune sites as in the case of the forest fire model. Furthermore, the set of immune sites and the set of infected sites may both the quite disconnected. However, they are still referred to as clusters here.

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This letter considers the time development of the cluster of immune sites and the cluster of infected sites, starting from a central infected site, for a range of values p, r. The lattice is triangular of size $L \times L$ with $L = 199$ in all but a few instances. The number of time steps ranges from 125 to 500 such that infections never reach the lattice edge, avoiding edge effects. For each value of p less than the standard percolation threshold p_c, a critical value of r is found such that for r greater than this value the number of infected sites continues to increase with time. Two phase regions can immediately be identified. In one, the spreading of infections is finite, while in the other the spreading is infinite. Figure 1 shows the percolating and non-percolating phase regions separated by the line of percolation threshold values. At $r = 0$, $p_c = 0.347\,29$ corresponds to ordinary percolation.

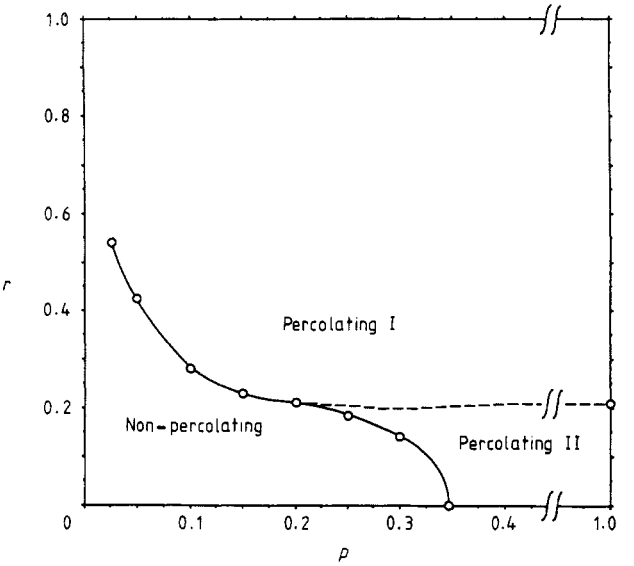

Figure 1. Phase diagram of the imperfect immunisation process for the triangular lattice. p and r represent infection and re-infection probabilities, respectively. The broken curve is the expected form of the threshold between the tricritical point and the calculated critical point at $p = 1$.

Just above threshold, the number (mass) of either infected or immune sites is expected to increase with time as a power law

$$M \sim t^{\alpha}.$$

The radius of gyration of the cluster of sites increases similarly,

$$R_g \sim t^{\beta}.$$

The fractal dimensionality d_f is defined by

$$M \sim R_g^{d_f}$$

so $d_f = \alpha/\beta$; for a complete definition of the exponents and reference to the original literature, see [12].

Figure 2 shows the fractal dimensionality d_f just above threshold for various values of (p, r), with p specified. The immune cluster is characterised by $d_f \simeq 2$ over the whole

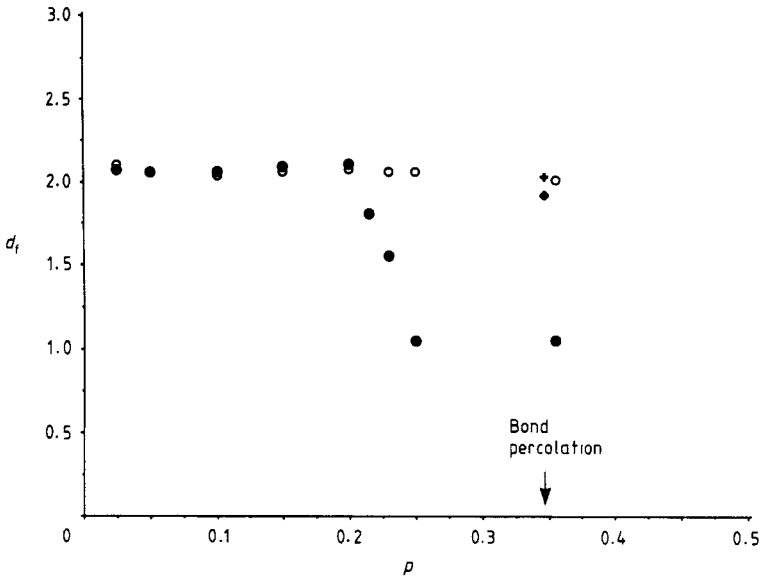

Figure 2. Fractal dimensionality d_f of the cluster of immune sites (○) and infected sites (●) for the imperfect immunisation process just above the percolation threshold, for various critical values of p. Also shown are Grassberger's result (◆) and von Niessen and Blumen's value (+) for the fractal dimensionality of the immune cluster just above the standard percolation threshold.

range of (p, r). The fractal dimensionality of the immune cluster just above the standard percolation threshold ($p = p_c$, $r = 0$), $d_f = 2.00$, is compared with that calculated from Grassberger's results for a square lattice, $d_f = 1.90$ [15] and von Niessen and Blumen's calculation for the triangular lattice, $d_f = 2.01$ [6]. Our result, like von Niessen and Blumen's is based on viewing the actual time development of the cluster. For $p \leqslant 0.21$ ($r \leqslant 0.21$), the infected cluster exhibits similar behaviour to that of the immune cluster. On the other hand, for larger p (smaller r), the fractal dimensionality for the infected cluster drops sharply to its value just above the standard percolation threshold, $d_f = 1.05$.

One knows, for example, that for $p = 1$, $r = 0$ the infections form a simple expanding ring with $d_f \simeq 1.1$ for the triangular lattice. On the other hand, $p = 1$, $r = 1$ leads to alternating infected and immune rings for which $d_f = 2$. This suggests that a lower d_f for an infected site reflects fewer infections in the interior of the infected-immune network which relies on the reinfection process to sustain the infected population. At some intermediate value of r, there is a transition from a percolating phase characterised by infections primarily in the vicinity of the (expanding) perimeter of the infected-immune region to a percolating phase with infected sites situated throughout the cluster.

It is just such a transition from one type of percolating phase to another which is described in figure 2 by the sharp drop in fractal dimensionality of the infected cluster for the triangular lattice. The point $p = r = 0.21$ is the junction of all three phase regions, one non-percolating and the two percolating phases, called the tricritical point. Figure 1 also gives the point on the line of percolation threshold values at which $p = r$ as 0.21.

For $p = 1$, the primary infections form an unbroken expanding ring, so that each site is infected in turn and all the interior sites are either immune (as they recover) or infected (as they are re-infected). Far away from the centre of this expanding network,

the initial infected site, the infection front can be viewed as linear (part of one edge of a hexagon). A particular line of sites becomes, first, a line of infection and, in subsequent time steps, a line of immune sites and, finally, a mix of immune and infected sites as re-infection takes place from both sides. If the re-infection probability r is less than the threshold value, the number of infected sites on this and any other line will tend to zero with time. Periodic boundary conditions are imposed in the transverse direction on a lattice of 49×201 sites. Only infections from the side of the receding infection front are tabulated for reasons of efficiency.

The threshold value for the reinfection probability r at $p = 1$ is determined from figure 3 to be 0.210 ± 0.002, suggesting that the threshold between one type of percolating phase and the other is the same at $p = 1$ as for the no-immunisation case, $p = r$, as suggested by Ohtsuki and Keyes [16]. Since, for $p = 1$, all sites in their turn become infected and then immune, the subsequent time development of these sites depends only on the re-infection probability r. This is equally true for the no-immunisation case where the infection probability p and reinfection probability r are, in fact, identical. An infected population is sustained within the cluster only if r is greater than the percolation threshold and this threshold is the same for the two cases.

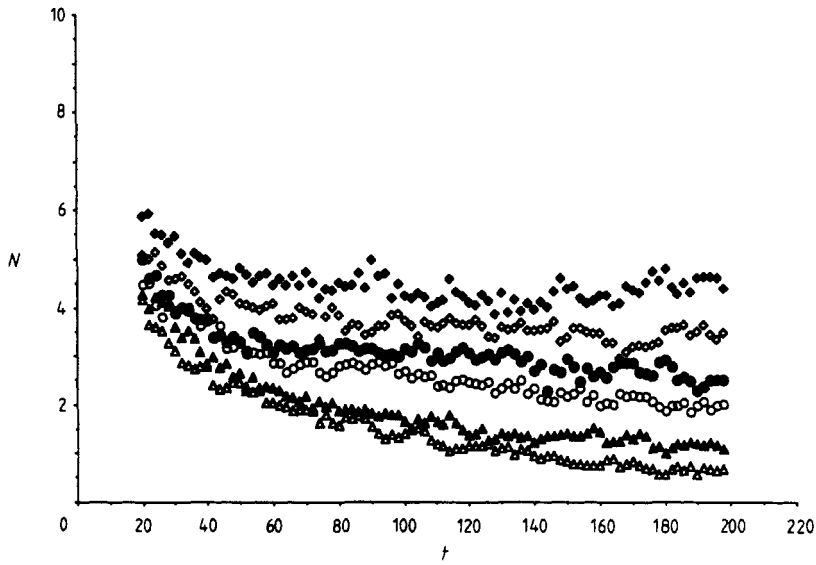

Figure 3. The number of infections as a function of time t along a line of sites behind a linear infection front for $p = 1$ and $r = 0.212$ (◆), $r = 0.210$ (◇), $r = 0.208$ (●), $r = 0.206$ (○), $r = 0.204$ (▲), $r = 0.202$ (◇).

A network consisting of immune and infected sites exhibits, for $r = 1$, behaviour similar to that of blinking Christmas tree lights. Each site alternates indefinitely between the immune and the infected state. After an even number of time steps the number of infected (immune) sites is unchanged. In fact, it is the same set of infected (immune) sites. Simulations for $r < 1$ do not provide evidence of this sort of long-term 'memory'. Rather, the infections (immune sites) tend towards a random distribution.

Simulations of an epidemic process with imperfect immunisation on a triangular lattice confirm the existence of three phase regions: one non-percolating and two percolating. The tricritical point, corresponding to the junction of the three regions

occurs at $p = r = 0.21$ for the triangular lattice. The sharp drop in the fractal dimensionality of the infected cluster just above the percolation threshold clearly distinguishes between the two percolating phases. The implication is that for lower values of the reinfection probability r, infections are limited to the vicinity of the perimeter. The transition between the percolating phases for $p = 1$ is shown to also occur at $r = 0.21$.

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References

[1] Alexandrowicz Z 1980 *Phys. Lett.* **80A** 284
[2] Cardy J L 1983 *J. Phys. A: Math. Gen.* **16** L709
[3] Grassberger P 1983 *Math. Biosci.* **63** 157
[4] Cardy J L and Grassberger P 1985 *J. Phys. A: Math. Gen.* **18** L267
[5] MacKay G and Jan N 1984 *J. Phys. A: Math. Gen.* **17** L757
[6] Von Niessen W and Blumen A 1986 *J. Phys. A: Math. Gen.* **19** L289
[7] Ohtsuki T and Keyes T 1986 *J. Phys. A: Math. Gen.* **19** L281
[8] Kerstein A R 1984 *Phys. Rev.* B **30** 2980; 1985 *Phys. Rev.* B **31** 321
[9] Family F and Landau D P 1984 (ed) *Kinetics of Aggregation and Gelation* (Amsterdam: North-Holland)
[10] Wilkinson D and Willemsen J F 1983 *J. Phys. A: Math. Gen.* **16** 3365
[11] Schulman L S and Seiden P E 1983 *Percolation Structures and Processes* (*Annals Israel Phys. Soc.* **5**) ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 251
[12] Stanley H E and Ostrowsky N (ed) *On Growth and Form* (Dordrecht: Martinus Nijhoff)
[13] Mollison D 1977 *J. R. Stat. Soc.* B **39** 283
[14] Bailey N T J 1975 *Mathematical Theory of Infectious Diseases* (London: Griffin)
[15] Grassberger P 1985 *J. Phys. A: Math. Gen.* **18** L215
[16] Ohtsuki T and Keyes T 1986 *Phys. Rev.* A **33** 1223